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Symmetric and unitary group representations: I. Duality theory

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Abstract. An extension of the Schur–Weyl duality connecting the representations of the symmetric and unitary groups is given. The Schur–Weyl basis is constructed using annihilation and creation operators. Three factorisation lemmas are derived. Their importance lies in the fact that they relate the phase freedoms within the Racah–Wigner algebra of the symmetric groups and the unitary groups. Extensions of the Regge symmetries are also given. These are expressed in five duality relations.

1. Introduction

The connection between the symmetric and unitary groups has been known since the work of Schur and Frobenius. Later, Weyl (1931, 1946) showed that the Young symmetrisers developed for the symmetric groups may be used to obtain the irreducible representations (irreps) of the unitary groups (see also Murnaghan 1938).

Weyl used this duality, gave numerous theorems concerned with irreps of both groups, and also gave applications to the many-body system of f equivalent particles. Such systems arise in many areas from molecular physics to elementary particle physics.

The Schur function approach, however, makes the duality more apparent. These functions (Schur 1901, Littlewood 1940) had been studied by Jacobi, Trudi, Kostka and others under the name of bialternants long before Schur showed their connection with the characters of the symmetric and unitary groups. The use of the purely combinatoric properties of Schur functions is still proving fruitful in obtaining new identities and thus new computational techniques for character theory (see King 1970, Wybourne 1970, Butler and King 1973a, b, King *et al* 1981, Black *et al* 1983).

The duality goes further than that expressed by the Schur functions. Many powerful equalities between various transformation factors of the symmetric groups and those of the unitary groups can be established.

Jahn (1950) was the first of many nuclear shell model theorists to use the duality to compute the *jm* and *j* symbols of a unitary group, work which was later much extended (Jahn 1954, Elliott *et al* 1953, Kaplan 1962a, b, Horie 1964, Kramer and Seligman 1969b, Vanagas 1971). Results are derived using the Young symmetrisers of the symmetric group as projectors for the unitary group.

Kramer (1967) used explicit transformations between the bases defined in terms of different symmetric group chains to define his f symbol (our resubduction factor) for a symmetric group. He showed that the f symbols were essentially equivalent to recoupling coefficients ($6j$ and $9j$ symbols) for any unitary group (Kramer 1968) and further that f symbols were also equal to coupling coefficients ($3jm$ symbols) for $U_{p+q} \supset U_p \times U_q$. The symmetry properties of the symmetric group f symbol together with the duality result gave the origin of the Regge symmetries for the $6j$ symbols of SU_2, and for the $3jm$ symbols of $SU_2 \supset U_1$ (equivalently $SO_3 \supset SO_2$) (Kramer and Seligman 1969a).

A simpler formulation of the various transformations followed using the concept of double coset (DC) generators of the symmetric group (Kramer and Seligman 1969b). Sullivan (1973, 1975a, b, 1976, 1978a, b, 1980) has formulated the general theory of DC decompositions developing many more duality results.

In this paper we further extend the Schur–Weyl duality. The group theory and transformation theory that we require have been given in a previous paper (Haase and Butler (1984). Section 2 presents a construction of the Schur–Weyl basis using creation and annihilation operators. Three factorisation lemmas are derived in § 3. Arising in these lemmas are three 'symmetric group – unitary group duality factors' which have been omitted or assumed to be unity by previous authors. The importance of these factors lies in the fact that they relate the phase and multiplicity freedoms within the Racah–Wigner algebra of the symmetric groups to similar freedoms for the unitary groups.

For some phase choices these duality factors are not unity. One of several important topics is the distinction between U_p and SU_p. The duality relations of Kramer and Seligman, and of Sullivan, are derived directly from our lemmas in § 4. The relations give extensions of the Regge symmetries of the SU_2 $6j$ symbols and the $SU_2 \supset U_1$ $3jm$ symbols, to all unitary groups.

2. The Schur–Weyl basis

The dual structures of the symmetric and unitary groups may be exhibited in the language of creation and annihilation operators (Jordan 1935, Schwinger 1952, Baird and Biedenharn 1963, Moshinsky 1963). One constructs a Hilbert space to carry representations of the symmetric group S_f and the unitary group U_p. Lezuo (1972) has used such a realisation to study $S_f \times U_3$. The creation operator formulation makes the Schur–Weyl duality quite apparent.

In this 'second quantisation' notation, the single-particle basis states are given by boson (or fermion) creation operators acting on a suitably defined vacuum state $|0\rangle$

$$a_k^\dagger|0\rangle \qquad (1 \leq k \leq p). \tag{2.1}$$

These operators have the usual commutation (or anticommutation) relations

$$a_k \equiv (a_k^\dagger)^\dagger, \qquad [a_k^\dagger, a_l^\dagger]_\mp = 0 = [a_k, a_l]_\mp, \tag{2.2}$$

$$[a_k^\dagger, a_l]_\mp = \delta_{kl}. \tag{2.3}$$

Using these basic relations the p^2 operators,

$$F_{kl} \equiv a_k^\dagger a_l, \qquad 1 \leq k, l \leq p, \tag{2.4}$$

are found to satisfy the commutators

$$[F_{kl}, F_{mn}] = \delta_{lm} F_{kn} - \delta_{kn} F_{lm}. \tag{2.5}$$

Hence the F_{kl} are closed under commutation and describe the Lie algebra of U_p. The

p basis states $a_k^\dagger|0\rangle$ transform as the defining irrep $\varepsilon_p \equiv \{1\}$ of U_p and we may write

$$a_k^\dagger|0\rangle = |\varepsilon_p k\rangle \qquad (k=1,\ldots,p). \tag{2.6}$$

The f-particle basis states are constructed by a tensor product of f-boson (or fermion) creation operators acting on the vacuum state

$$|0\rangle \equiv |0\rangle \ldots |0\rangle \qquad (f \text{ times}), \tag{2.7}$$

$$a_{k_1}^{\dagger 1} \ldots a_{k_f}^{\dagger f}|0\rangle, \tag{2.8}$$

where $a_{k_i}^{\dagger i}$ creates the ith particle in the basic state k_i ($1 \leqslant i \leqslant f$, $1 \leqslant k_i \leqslant p$).

These creation operators have similar properties to those of single-particle creation operators

$$a_k^i \equiv (a_k^{\dagger i})^\dagger, \tag{2.9}$$

$$[a_k^i, a_l^j]_\mp = 0 = [a_k^{\dagger i}, a_l^{\dagger j}]_\mp, \tag{2.10}$$

$$[a_k^{\dagger i}, a_l^j]_\mp = \delta_{ij}\delta_{kl}. \tag{2.11}$$

The p^f f-particle states transform according to the f-Kronecker product irreps $\varepsilon_p^f \equiv \varepsilon_p \times \ldots \times \varepsilon_p$ (f times) of $U_p^f \equiv U_p \times \ldots \times U_p$ (f times). We may thus label the states as

$$\begin{aligned} a_{k_1}^{\dagger 1} \ldots a_{k_f}^{\dagger f}|0\rangle &= |\varepsilon_p k_1\rangle \ldots |\varepsilon_p k_f\rangle \\ &\equiv |\varepsilon_p^f k_1 \ldots k_f\rangle, \qquad 1 \leqslant k_1, \ldots, k_f \leqslant p. \end{aligned} \tag{2.12}$$

A realisation of the generators of both U_p and S_f can be constructed from the creation and annihilation operators. The generators have well defined actions on all f-particle states. The set of p^2 operators

$$F_{kl} = \sum_{i=1}^{f} a_k^{\dagger i} a_l^i, \qquad 1 \leqslant k, l \leqslant p, \tag{2.13}$$

generate under commutation the Lie algebra of U_p, while the transposition operators

$$\tau_{ij} = \sum_{k,l} a_k^{\dagger i} a_l^{\dagger j} a_l^i a_k^j, \qquad 1 \leqslant i, j \leqslant f, \tag{2.14}$$

generate the symmetric group S_f. The f-Kronecker product space ε_p^f thus furnishes a p^f-dimensional representation space for both U_p and S_f.

Most importantly, since each operator of U_p in this realisation commutes with each operator of S_f, the space ε_p^f is a representation space for the direct product group $S_f \times U_p$, which we call the Schur–Weyl group. The standard result (Weyl 1931, Murnaghan 1938, Littlewood 1940) is that we have a unique decomposition of ε_p^f into subspaces which transform irreducibly under the action of the operators of the Schur–Weyl group. Each irrep of $S_f \times U_p$ in ε_p^f can be labelled

$$\lambda(S_f) \times \lambda'(U_p) \tag{2.15}$$

where λ is a partition of f into not more than p parts, $(\lambda) = (\lambda_1\lambda_2 \ldots \lambda_p)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p \geqslant 0$ and $\lambda_1 + \lambda_2 + \ldots \lambda_p = f$. The result central to the duality is that each irrep $\lambda(S_f)$ occurs with a unique irrep $\lambda'(U_p)$ and *vice versa*. The representation labels of symmetric and unitary groups are usually chosen so that this uniqueness is emphasised, i.e. by using the same partition $\lambda' = \lambda$. The occurrence of each irrep $\lambda(S_f) \times \lambda(U_p)$ is multiplicity free. Hence we have the following transformation of basis for the space ε_p^f,

$$|\varepsilon_p^f k_1 \ldots k_f\rangle = \sum_{\lambda i l} |\varepsilon_p^f \lambda i \lambda l\rangle\langle\varepsilon_p^f \lambda i \lambda l|\varepsilon_p^f k_1 \ldots k_f\rangle \tag{2.16}$$

where i (respectively l) labels the basis of irrep space λ of S_f (respectively U_p). An explicit reduction may be obtained via the application of Young symmetrisers.

The action of the operators $\tau \times F$ in their representation of $S_f \times U_p$ on this basis, which we call the Schur–Weyl basis, is given by

$$\tau \times F|\varepsilon_p^f \lambda i \lambda l\rangle = |\varepsilon_p^f \lambda i' \lambda l'\rangle \lambda(\tau)^{i'}{}_i \lambda(F)^{l'}{}_l \tag{2.17}$$

where $\lambda(\tau)^{i'}{}_i$ and $\lambda(F)^{l'}{}_l$ are elements of a standard irreducible matrix representation λ of S_f and U_p respectively. For convenience we write the Schur–Weyl basis as

$$|\varepsilon_p^f \lambda i \lambda l\rangle \equiv \left| \varepsilon_p^f \quad \begin{matrix} i \\ \lambda \\ l \end{matrix} \right\rangle. \tag{2.18}$$

No choice of basis within the irrep spaces of either S_f or U_p is implied in the above. Of course, special bases do exist for both groups. The most important are known as the Young–Yamanouchi basis ($S_f \supset S_{f-1} \times S_1 \supset S_{f-2} \times S_1 \times S_1 \ldots$) for the symmetric groups and the Weyl–Gel'fand basis ($U_p \supset U_{p-1} \times U_1 \supset U_{p-2} \times U_1 \times U_1 \supset \ldots$) for the unitary groups. The latter has been used extensively by both Moshinsky and Biedenharn and their several collaborators. In the following we obtain the results that are valid for bases chosen with respect to subgroups that are direct product groups of a less restricted nature.

3. Transformation factors for three group–subgroup chains

We produce three types of transformation which take the Schur–Weyl basis states $|\varepsilon_p^f \lambda i \lambda l\rangle$ into one of the following group–subgroup schemes:

(1) The dissociation of the space ε_p^f into the direct product of $\varepsilon_p^{f_1}$ with $\varepsilon_p^{f_2}$ ($f = f_1 + f_2$).

(2) The transformation $\varepsilon_p^f \to \varepsilon_{p_1}^f \times \varepsilon_{p_2}^f$ ($p = p_1 p_2$) obtained by the reduction $\varepsilon_p \to \varepsilon_{p_1} \times \varepsilon_{p_2}$ in $U_p \supset U_{p_1} \times U_{p_2}$.

(3) The transformation $\varepsilon_p^f \to \bigoplus_t \binom{f}{t} \varepsilon_q^t \times \varepsilon_{\bar{q}}^{\bar{t}}$ ($\bar{q} = p - q$, $\bar{t} = f - t$) obtained by the reduction $\varepsilon_p \to \varepsilon_q \cdot 0_{\bar{q}} + 0_q \cdot \varepsilon_{\bar{q}}$ in $U_p \supset U_q \times U_{\bar{q}}$ where $\binom{f}{t} = f!/(t!\,f-t!)$ is the multiplicity of $\varepsilon_q^t \times \varepsilon_{\bar{q}}^{\bar{t}}$.

The uniqueness of the Schur–Weyl basis determines three transformation factors which we will call duality factors. The numerical values of these factors depend only on the phase and multiplicity choices within the Racah–Wigner algebra of the symmetric and unitary groups (Haase 1983).

Consider the first group–subgroup scheme depicted in figure 1. The irrep space ε_p^f is isomorphic to the direct product of $\varepsilon_p^{f_1}$ and $\varepsilon_p^{f_2}$ with $f = f_1 + f_2$. Each Kronecker product space is decomposed to its corresponding Schur–Weyl basis. The subgroup $S_{f_1} \times S_{f_2} \times U_p$ is obtained by the subduction $S_f \supset S_{f_1} \times S_{f_2}$ ($f = f_1 + f_2$) on the left side of figure 1 and by the coupling $U_p \times U_p \supset U_p$ on the right side of figure 1. Both the coupling and subduction processes are given by the outer multiplication of Schur

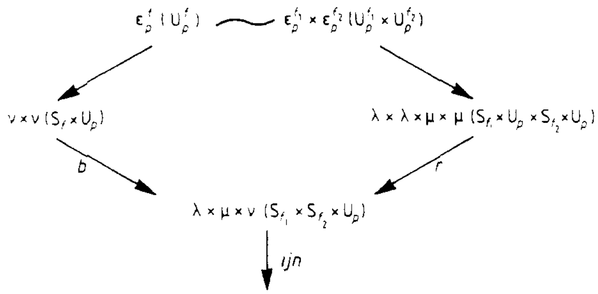


Figure 1. Where $f = f_1 + f_2$.

functions (the Littlewood–Richardson rule (Littlewood 1940, p 94))

$$\{\lambda\} \times \{\mu\} = \sum m^{\nu}_{\lambda\mu}\{\nu\}. \tag{3.1}$$

It is well known (Weyl 1931, theorem 3, p 339) that if the representation $\lambda \times \mu$ of U_p contains the irrep ν exactly $m^{\nu}_{\lambda\mu}$ times then conversely the irrep ν of S_f contains, on subduction to $S_{f_1} \times S_{f_2}$, the irrep $\lambda \times \mu$ exactly $m^{\nu}_{\lambda\mu}$ times.

Comparing figure 1 with figure 1 of Haase and Butler (1984), we find that the following lemma is just an application of (2.14) and (2.15) of that paper.

Lemma 1. The duality factor of figure 1 is given by

$$\left| \begin{matrix} & ij \\ \varepsilon_p^{f_1}\varepsilon_p^{f_2} & \lambda\mu \\ & r\nu n \end{matrix} \right\rangle = \left| \begin{matrix} & b\lambda i\mu j \\ \varepsilon_p^{f} & \nu \\ & n \end{matrix} \right\rangle D_p(\lambda\mu, \nu)^b{}_r \tag{3.2}$$

where we have written

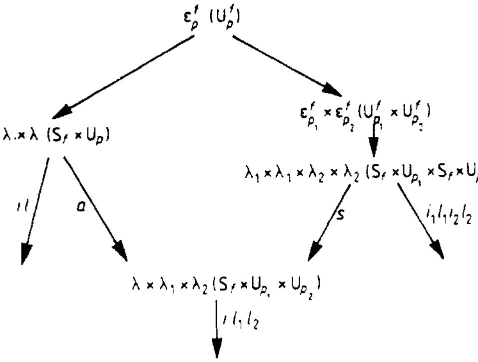
$$D_p(\lambda\mu, \nu)^b{}_r \equiv \left\langle \begin{matrix} & b\lambda\mu \\ \varepsilon_p^{f} & \nu \\ & \end{matrix} \middle| \begin{matrix} \varepsilon_p^{f_1}\varepsilon_p^{f_2} & \lambda\mu \\ & r\nu \end{matrix} \right\rangle. \tag{3.3}$$

The duality factor is an element of a square matrix of dimension $m^{\nu}_{\lambda\mu}$. This number depends on the partitions λ, μ, ν and the group orders f_1, f_2, f, p, and is given by the Littlewood–Richardson rule.

Our second duality factor is obtained by considering the transformation between the bases of $U_p^f \supset S_f \times U_p \supset S_f \times U_{p_1} \times U_{p_2}$ and $U_p^f \supset U_{p_1}^f \times U_{p_2}^f \supset S_f \times U_{p_1} \times S_f \times U_{p_2} \supset S_f \times U_{p_1} \times U_{p_2}$ (figure 2). The first basis involves the subduction λ to $\lambda_1 \times \lambda_2$ of $U_p \supset U_{p_1} \times U_{p_2}$ with $p_1 p_2 = p$ while the second couples $\lambda_1 \times \lambda_2$ to λ in S_f. Both processes are given by the inner multiplication of Schur functions,

$$\{\lambda_1\} \circ \{\lambda_2\} = \sum_{\lambda} g^{\lambda}_{\lambda_1\lambda_2}\{\lambda\}.$$

Applying (2.14) and (2.15) of Haase and Butler (1984) to figure 2 w derive the following lemma.



11017 **Figure 2.** Where $p = p_1 p_2$.

Lemma 2. The duality factor of figure 2 is given by

$$\left| \begin{matrix} & i \\ \varepsilon_p^f & \lambda \\ & a\lambda_1 l_1 \lambda_2 l_2 \end{matrix} \right\rangle = \left| \begin{matrix} & s\lambda i \\ \varepsilon_{p_1}^f \varepsilon_{p_2}^f & \lambda_1 \lambda_2 \\ & l_1 l_2 \end{matrix} \right\rangle D_{p_1 p_2}(\lambda, \lambda_1 \lambda_2)^s{}_a \tag{3.4}$$

where we use the notation

$$D_{p_1 p_2}(\lambda, \lambda_1 \lambda_2)^s{}_a \equiv \left\langle \begin{matrix} & s\lambda \\ \varepsilon_{p_1}^f \varepsilon_{p_2}^f & \lambda_1 \lambda_2 \\ & \end{matrix} \right| \left. \begin{matrix} & \\ \varepsilon_p^f & \lambda \\ & a\lambda_1 \lambda_2 \end{matrix} \right\rangle . \tag{3.5}$$

This duality factor is also an element of a square matrix of dimension $g_{\lambda_1 \lambda_2}^{\lambda}$ given by the inner multiplication of Schur functions. This factor depends only on the partitions λ, λ_1, λ_2 and group orders f, p, p_1, p_2. The label f is implicit in each λ, λ_1, λ_2 since each must be a partition of f.

The third duality factor is obtained by the reduction (figure 3) $\varepsilon_p \to \varepsilon_q \cdot 0_{\bar{q}} + 0_q \cdot \varepsilon_{\bar{q}}$ under $U_p \supset U_q \times U_{\bar{q}}$ where $\bar{q} = p - q$ and 0_q, $0_{\bar{q}}$ are the identity irreps of U_q and $U_{\bar{q}}$ respectively. The f-Kronecker product space $\varepsilon_p^f \to (\varepsilon_q \cdot 0_{\bar{q}} + 0_q \cdot \varepsilon_{\bar{q}})^f$ is expanded as a direct sum of direct product spaces $\varepsilon_q^t \times \varepsilon_{\bar{q}}^{\bar{t}}$. That is, we have

$$(\varepsilon_q \cdot 0_{\bar{q}} + 0_q \cdot \varepsilon_{\bar{q}})^f = \bigoplus_{t=0}^{f} \binom{f}{t} \varepsilon_q^t \times \varepsilon_{\bar{q}}^{\bar{t}} \tag{3.6}$$

where for fixed t, the multiplicity of $\varepsilon_q^t \times \varepsilon_{\bar{q}}^{\bar{t}}$ in ε_p^f is $\binom{f}{t} = f!/t!\bar{t}!$. After reducing each Kronecker product group to its corresponding Schur–Weyl group, (see figure 3) we can then perform for each t the induction $S_t \times S_{\bar{t}}$ into S_f. This step can be understood by recognising that the basis vectors $|\varepsilon_p^f r_t \varepsilon_q^t \mu j \mu m \varepsilon_{\bar{q}}^{\bar{t}} \nu k \nu n\rangle$ ($r_t = 1, \ldots, \binom{f}{t}$), for fixed $U_q \times U_{\bar{q}}$ basis vector labels $(\mu m \nu n)$ and varying $(r_t j k)$ labels, are the basis vectors of the induced representation $\mu \times \nu$ of $S_t \times S_{\bar{t}}$ in S_f. (This replication of $S_t \times S_{\bar{t}}$ is the reason for placing r_t in figure 3 as a basis label of $S_t \times U_q \times S_{\bar{t}} \times U_{\bar{q}}$ rather than a branching multiplicity label of $U_p^f \supset U_q^t \times U_{\bar{q}}^{\bar{t}}$.) The induced representation space is reducible in S_f; its constituent irreps are given by the outer multiplication of Schur functions

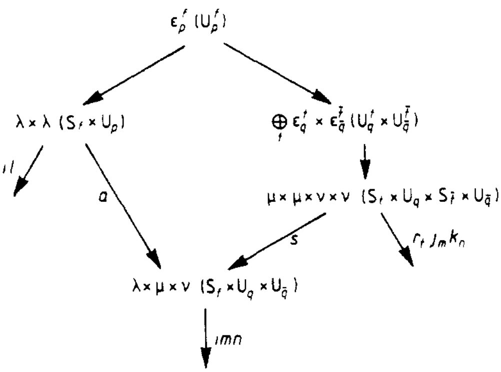
$$\{\mu\}\times\{\nu\}=\sum_{\lambda} m^{\lambda}_{\mu\nu}\{\lambda\}. \tag{3.7}$$

The dimension of the induced space is (Wybourne 1970, equation (45))

$$(f!/t!\bar{t}!)|\mu|_{S_t}\times|\nu|_{S_{\bar{t}}}=\sum_{\lambda} m^{\lambda}_{\mu\nu}|\lambda|_{S_f}. \tag{3.8}$$

An explicit construction of the induced representation space can be obtained using coset theory (Coleman 1966, 1968, Bradley and Cracknell 1972) but this is not necessary in our approach (see Haase and Butler 1984).

An alternative labelling of basis vectors is given by the GH basis $U^f_p\supset S_f\times U_p\supset S_f\times U_q\times U_{\bar{q}}$. The outer multiplication of Schur functions also determines the $U_p\supset U_q\times U_{\bar{q}}$ reduction. The application of Haase and Butler (1984, equations (2.14) and (2.15)) to figure 3 provides the third lemma.



11019 **Figure 3.** Where $p=q+\bar{q}$, $f=t+\bar{t}$.

Lemma 3. The duality factor of figure 3 is defined as

$$\left|\begin{matrix}\varepsilon^f_p & \begin{matrix} i \\ \lambda \\ a\mu m\nu n\end{matrix}\end{matrix}\right\rangle=\left|\begin{matrix}\varepsilon^f_p\varepsilon^t_q\varepsilon^{\bar{t}}_{\bar{q}} & \begin{matrix}\uparrow s\lambda i \\ \mu\nu \\ mn\end{matrix}\end{matrix}\right\rangle D_{q+\bar{q}}(\lambda,\mu\nu)^s{}_a \tag{3.9}$$

where

$$D_{q+\bar{q}}(\lambda,\mu\nu)^s{}_a\equiv\left\langle \varepsilon^f_p\varepsilon^t_q\varepsilon^{\bar{t}}_{\bar{q}}\,\begin{matrix}\uparrow s\lambda \\ \mu\nu\end{matrix}\right|\left.\varepsilon^f_p\,\begin{matrix}\lambda \\ a\mu\nu\end{matrix}\right\rangle \tag{3.10}$$

is an element of a square matrix depending on partition labels λ, μ, ν and group orders f, q, $\bar{q}$, p. It is of dimension $m^{\lambda}_{\mu\nu}$ given by (3.1).

We have defined three different duality factors which can be seen to connect the Racah–Wagner algebra multiplicity phase freedoms of the symmetric and unitary groups. Another way to interpret these lemmas is that they describe how to change any particular choice of unitary group branching and product multiplicity phases into a particular symmetric group multiplicity choice. Section 4 develops the powerful equalities between various transformations of the symmetric group and those of the unitary groups.

4. $S_f \times U_p$ duality results

In this section we derive five duality relations giving the precise relationship between unitary group transformation factors and corresponding symmetric group transformation factors. The importance of these duality results is illustrated in the numerical equivalence via duality factors of an infinite number of transformation factors of different unitary groups. These relations include the extension of the Regge symmetries of SU_2 and $SU_2 \supset U_1$ to symmetries for all unitary groups. After each statement of a duality relation we give an outline of its derivation.

Relation 1. The resubduction factor of the symmetric group scheme of figure 4(*a*) equals to within four duality factors a recoupling factor of U_p (figure 4(*b*)).

$$\langle \lambda b_1 \mu (b_2 \rho\sigma)\tau | \lambda b_3 \rho\nu (b_4 \sigma\tau)\rangle_{f_1+f_2+f_3}$$
$$= D_p(\rho\sigma, \mu)^{b_2}{}_{r_2} D_p(\mu\tau, \lambda)^{b_1}{}_{r_1} \langle (\rho\sigma) r_2 \mu, \tau, r_1 \lambda | \rho(\sigma\tau) r_4 \nu, r_3 \lambda\rangle_p \qquad (4.1)$$
$$\times D_p(\sigma\tau, \nu)^{\dagger r_4}{}_{b_4} D_p(\rho\nu, \lambda)^{\dagger r_3}{}_{b_3}.$$

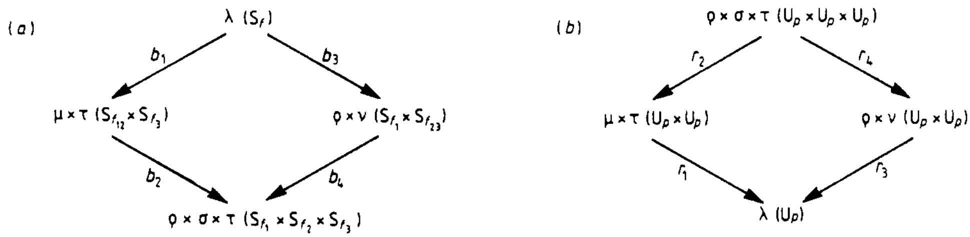


Figure 4. Where $f = f_{12} + f_3 = f_1 + f_{23} = f_1 + f_2 + f_3$.

The proof of this equality is obtained by considering the overlap of the two basis vectors,

$$\left| \varepsilon_p^{f_1} \varepsilon_p^{f_2} \varepsilon_p^{f_3} \begin{matrix} ijk \\ \rho\sigma\tau \\ r_2\mu, r_1\lambda l \end{matrix} \right\rangle \quad \text{and} \quad \left| \varepsilon_p^{f_1} \varepsilon_p^{f_2} \varepsilon_p^{f_3} \begin{matrix} ijk \\ \rho\sigma\tau \\ r_4\nu, r_3\lambda l \end{matrix} \right\rangle. \qquad (4.2)$$

This produces the unitary group recoupling factor (see Haase and Butler 1984, equation (2.19)). Using lemma 1 four times, the couplings in U_p are replaced by symmetric group subductions. The product multiplicity labels r_1, r_2, r_3, r_4 are changed to symmetric group branching multiplicity labels b_1, b_2, b_3, b_4. The overlap between the resulting basis vectors,

$$\left| \varepsilon_p^f \begin{matrix} b_1\mu(b_2\rho i\sigma j)\tau k \\ \lambda \\ l \end{matrix} \right\rangle \quad \text{and} \quad \left| \varepsilon_p^f \begin{matrix} b_3\rho i\nu(b_4\sigma j\tau k) \\ \lambda \\ l \end{matrix} \right\rangle, \qquad (4.3)$$

produces the $S_{f_1+f_2+f_3}$ resubduction factor (see Haase and Butler 1984, equation (2.21)).

Relation 2. A recoupling factor of S_f, figure 5(a), equals to within certain duality factors a resubduction factor of the unitary group scheme of figure 5(b).

$$\begin{aligned}\langle(\lambda_1\lambda_2)s_{12}\lambda_{12},\ \lambda_3,\ s\lambda|\lambda_1(\lambda_2\lambda_3)s_{23}\lambda_{23},\ s'\lambda\rangle_f \\ = D_{p_{12}p_3}(\lambda,\ \lambda_{12}\lambda_3)^{s}{}_{a}D_{p_1p_2}(\lambda_{12},\ \lambda_1\lambda_2)^{s_{12}}{}_{a_{12}} \\ \times\langle\lambda a\lambda_{12}(a_{12}\lambda_1\lambda_2)\lambda_3|\lambda a'\lambda_1\lambda_{23}(a_{23}\lambda_2\lambda_3)\rangle_{p_1p_2p_3} \\ \times D_{p_1p_{23}}(\lambda,\ \lambda_1\lambda_{23})^{\dagger a'}{}_{s'}D_{p_2p_3}(\lambda_{23},\ \lambda_2\lambda_3)^{\dagger a_{23}}{}_{s_{23}}.\end{aligned} \tag{4.4}$$

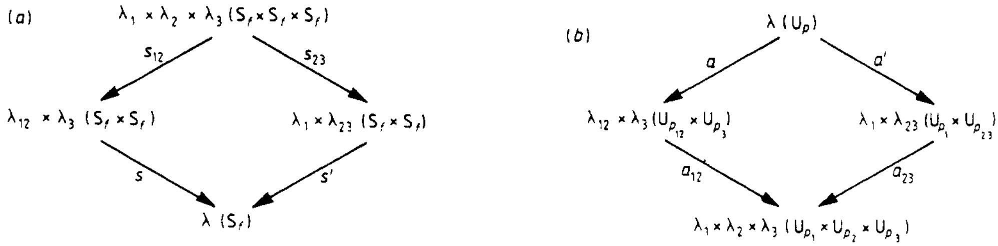


Figure 5. Where $p = p_{12}p_3 = p_1p_{23} = p_1p_2p_3$.

The proof is obtained by considering the overlap between the basis vectors

$$\left|\begin{matrix} & i \\ \varepsilon^f_p & \lambda \\ & a\lambda_{12}(a_{12}\lambda_1 l_1\lambda_2 l_2)\lambda_3 l_3\end{matrix}\right\rangle \quad\text{and}\quad \left|\begin{matrix} & i \\ \varepsilon^f_p & \lambda \\ & a'\lambda_1 l_1\lambda_{23}(a_{23}\lambda_2 l_2\lambda_3 l_3)\end{matrix}\right\rangle. \tag{4.5}$$

This gives the unitary group $U_{p_1p_2p_3}$ resubduction factor (Haase and Butler 1984, (2.21)). Using lemma 2, each of the unitary group branching labels a_{12}, a_{23}, a, a' can then be replaced by symmetric group product labels s_{12}, s_{23}, s. s'. The S_f recoupling factor is determined from the overlap of the resulting basis vectors

$$\left|\begin{matrix} & s_{12}\lambda_{12}s\lambda i \\ \varepsilon^f_{p_1}\varepsilon^f_{p_2}\varepsilon^f_{p_3} & \lambda_1\lambda_2\lambda_3 \\ & l_1l_2l_3\end{matrix}\right\rangle \quad\text{and}\quad \left|\begin{matrix} & s_{23}\lambda_{23}s'\lambda i \\ \varepsilon^f_{p_1}\varepsilon^f_{p_2}\varepsilon^f_{p_3} & \lambda_1\lambda_2\lambda_3 \\ & l_1l_2l_3\end{matrix}\right\rangle \tag{4.6}$$

(see Haase and Butler 1984, equation (2.19)).

Relation 3. A reinduction factor of the symmetric group scheme for figure 6(a) equals to within certain duality factors a resubduction factor of the unitary group scheme of figure 6(b).

$$\begin{aligned}\langle(\rho\sigma)\uparrow s_1\mu,\ \tau,\ \uparrow s_2\lambda|\rho(\sigma\tau)\uparrow s_3\nu,\ \uparrow s_4\lambda\rangle_{f_1+f_2+f_3} \\ = D_{p_{12}+p_3}(\lambda,\ \mu\tau)^{s_2}{}_{a_2}D_{p_1+p_2}(\mu,\ \rho\sigma)^{s_1}{}_{a_1}\langle\lambda a_2\mu(a_1\rho\sigma)\tau|\lambda a_4\rho\nu(a_3\sigma\tau)\rangle_{p_1+p_2+p_3} \\ \times D_{p_1+p_{23}}(\lambda,\ \rho\nu)^{\dagger a_4}{}_{s_4}D_{p_2+p_3}(\nu,\ \sigma\tau)^{\dagger a_3}{}_{s_3}.\end{aligned} \tag{4.7}$$

The proof is initiated with the overlap of the basis vectors

$$\left|\begin{matrix} & i \\ \varepsilon^f_p & \lambda \\ & a_2\mu(a_1\rho l\sigma m)\tau n\end{matrix}\right\rangle \quad\text{and}\quad \left|\begin{matrix} & i \\ \varepsilon^f_p & \lambda \\ & a_4\rho l\nu(a_3\sigma m\tau n)\end{matrix}\right\rangle. \tag{4.8}$$

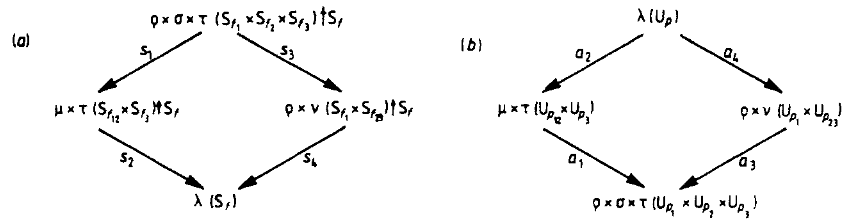


Figure 6. Where $(f=f_{12}+f_3=f_1+f_{23}=f_1+f_2+f_3)$ and $(p=p_{12}+p_3=p_1+p_{23}=p_1+p_2+p_3)$.

This defines the resubduction factor of the unitary group $U_{p_1+p_2+p_3}$ (Haase and Butler 1984, equation (2.21)). The unitary group branching multiplicity labels a_1, a_2, a_3, a_4 are replaced by branching multiplicity labels s_1, s_2, s_3, s_4 which are associated with symmetric group induction as determined by lemma 3. The overlap of the two resulting basis vectors

$$\left|\varepsilon^f_p\varepsilon^{f_1}_{p_1}\varepsilon^{f_2}_{p_2}\varepsilon^{f_3}_{p_3}\quad\begin{matrix}\uparrow s_1\mu,\uparrow s_2\lambda i\\ \rho\sigma\tau\\ lmn\end{matrix}\right\rangle \quad\text{and}\quad \left|\varepsilon^f_p\varepsilon^{f_1}_{p_1}\varepsilon^{f_2}_{p_2}\varepsilon^{f_3}_{p_3}\quad\begin{matrix}\uparrow s_3\nu,\uparrow s_4\lambda i\\ \rho\sigma\tau\\ lmn\end{matrix}\right\rangle \tag{4.9}$$

gives the reinduction factor of the symmetric group $(S_{f_1}\times S_{f_2}\times S_{f_3})\uparrow S_f$ (Haase and Butler 1984, equation (5.3)).

Relation 4. A coupling factor ($3jm$ symbol) of $S_f\supset S_{f_1}\times S_{f_2}$ (figure 7(a)) equals to within various duality factors a coupling factor ($3jm$ symbol) of $U_p\supset U_{p_1}\times U_{p_2}$ (figure 7(b)).

$$\left\langle\begin{matrix}\lambda_1\lambda_2\\ b_1b_2\\ \mu_1\nu_1\mu_2\nu_2\\ s_1s_2\\ \mu\nu\end{matrix}\;\middle|\;\begin{matrix}\lambda_1\lambda_2\\ s\\ \lambda\\ b\\ \mu\nu\end{matrix}\right\rangle^{f}_{f_1+f_2}$$
$$=D_{p_1p_2}(\mu,\mu_1\mu_2)^{s_1}{}_{a_1}D_{p_1p_2}(\nu,\nu_1\nu_2)^{s_2}{}_{a_2}D_{p_1}(\mu_1\nu_1,\lambda_1)^{b_1}{}_{r_1}D_{p_2}(\mu_2\nu_2,\lambda_2)^{b_2}{}_{r_2}$$
$$\times\left\langle\begin{matrix}\mu\nu\\ a_1a_2\\ \mu_1\mu_2\nu_1\nu_2\\ r_1r_2\\ \lambda_1\lambda_2\end{matrix}\;\middle|\;\begin{matrix}\mu\nu\\ r\\ \lambda\\ a\\ \lambda_1\lambda_2\end{matrix}\right\rangle^{p}_{p_1p_2}\quad D_p(\mu\nu,\lambda)^{\dagger r}{}_bD_{p_1p_2}(\lambda_1\lambda_2,\lambda)^{\dagger a}{}_s. \tag{4.10}$$

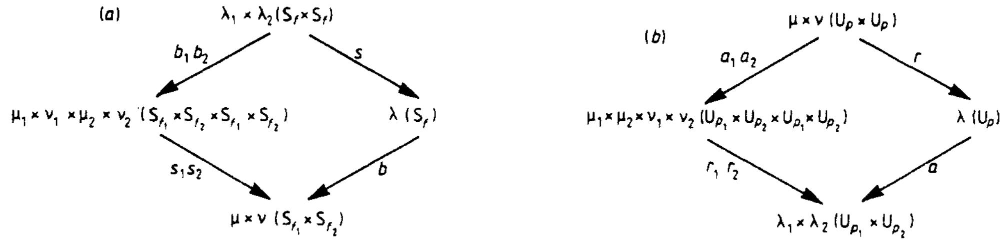


Figure 7. Where $(f=f_1+f_2)$ and $(p=p_1p_2)$.

Consider the scalar product between the two basis vectors

$$\left|\begin{matrix}\varepsilon_p^{f_1}\varepsilon_p^{f_2} & \begin{matrix} jk \\ \mu\nu \\ r\lambda a\lambda_1 l_1\lambda_2 l_2\end{matrix}\end{matrix}\right\rangle \quad \text{and} \quad \left|\begin{matrix}\varepsilon_p^{f_1}\varepsilon_p^{f_2} & \begin{matrix} jk \\ \mu\nu \\ a_1a_2\mu_1\mu_2\nu_1\nu_2r_1r_2\lambda_1l_1\lambda_2l_2\end{matrix}\end{matrix}\right\rangle. \tag{4.11}$$

This gives the $U_p \supset U_{p_1} \times U_{p_2}$ coupling factor. To each of these basis vectors we apply both lemma 1 and lemma 2. Each product multiplicity label of the unitary group becomes a symmetric group branching multiplicity label and *vice versa*. We then form the scalar product of the resulting basis vectors

$$\left|\begin{matrix}\varepsilon^f_{p_1}\varepsilon^f_{p_2} & \begin{matrix} s\lambda b\mu j\nu k \\ \lambda_1\lambda_2 \\ l_1 l_2\end{matrix}\end{matrix}\right\rangle \quad \text{and} \quad \left|\begin{matrix}\varepsilon^f_{p_1}\varepsilon^f_{p_2} & \begin{matrix} b_1b_2\mu_1\nu_1\mu_2\nu_2s_1s_2\mu j\nu k \\ \lambda_1\lambda_2 \\ l_1l_2\end{matrix}\end{matrix}\right\rangle \tag{4.12}$$

which gives the $S_f \supset S_{f_1} \times S_{f_2}$ coupling factor.

Relation 5. An induction factor of $S_{f_1} \times S_{f_2}$ to $S_t \times S_{\tilde{t}}$ (figure 8(*a*)) equals to within certain duality factors a coupling factor of $U_p \supset U_{p_1} \times U_{p_2}$ (figure 8(*b*))

$$\left\langle\begin{matrix} \mu\nu\uparrow \\ b_1b_2 \\ \rho\omega\sigma\zeta\uparrow \\ s_1s_2 \\ \tau\upsilon \end{matrix}\middle|\begin{matrix} \mu\nu\uparrow \\ s \\ \lambda \\ b \\ \tau\upsilon \end{matrix}\right\rangle^{f_1+f_2}_{t+\tilde{t}}$$

$$= D_{p_1+p_2}(\tau, \rho\sigma)^{s_1}{}_{a_1} D_{p_1+p_2}(\upsilon, \omega\zeta)^{s_2}{}_{a_2} D_{p_1}(\rho\omega, \mu)^{b_1}{}_{r_1} D_{p_2}(\sigma\zeta, \nu)^{b_2}{}_{r_2}$$

$$\times \left\langle\begin{matrix} \tau\upsilon \\ a_1a_2 \\ \rho\sigma\omega\zeta \\ r_1r_2 \\ \mu\nu \end{matrix}\middle|\begin{matrix} \tau\upsilon \\ r \\ \lambda \\ a \\ \mu\nu \end{matrix}\right\rangle_{p_1+p_2} \quad D_{p_1+p_2}(\tau\upsilon, \lambda)^{\dagger r}{}_b D_{p_1+p_2}(\lambda, \mu\nu)^{\dagger a}{}_s. \tag{4.13}$$

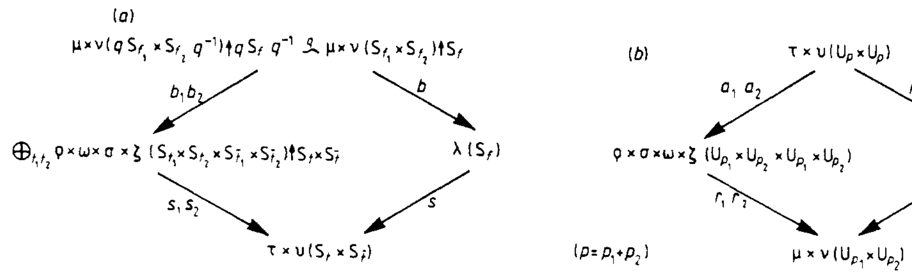


Figure 8.

The proof is given by taking the scalar product of the basis vectors,

$$\left|\begin{matrix}\varepsilon_p^{f_1}\varepsilon_p^{f_2} & \begin{matrix} kk' \\ \tau\upsilon \\ r\lambda a\mu m\nu n\end{matrix}\end{matrix}\right\rangle \quad \text{and} \quad \left|\begin{matrix}\varepsilon_p^{f_1}\varepsilon_p^{f_2} & \begin{matrix} kk' \\ \tau\upsilon \\ (a_1\rho\sigma, a_2\omega\zeta)r_1r_2\mu m\nu n\end{matrix}\end{matrix}\right\rangle \tag{4.14}$$

which defines the $U_{p_1+p_2} \supset U_{p_1} \times U_{p_2}$ coupling factor. Applying lemmas 1 and 3 to change multiplicity labels of the unitary groups to those of the symmetric group, we obtain the resulting basis vectors

$$\left| \varepsilon^{f}_{p_1+p_2} \varepsilon^{t}_{p_1} \varepsilon^{\bar{t}}_{p_2} \begin{matrix} \uparrow s\lambda b\tau k \upsilon k' \\ \mu\nu \\ mn \end{matrix} \right\rangle \quad \text{and} \quad \left| \varepsilon^{f}_{p_1+p_2} \varepsilon^{t}_{p_1} \varepsilon^{\bar{t}}_{p_2} \begin{matrix} (b_1\rho\omega, b_2\sigma\zeta) \uparrow s_1 s_2 \tau k \upsilon k' \\ \mu\nu \\ mn \end{matrix} \right\rangle . \tag{4.15}$$

The scalar product of these gives the induction factor for the symmetric group scheme of figure 8(a) (see Haase and Butler 1984, § 5).

The identification of recoupling factors of the unitary group with resubduction factors of the symmetric group (relation 1) was first demonstrated by Kramer (1968, equation 7.6)) for bipartition irreps of SU_2 and S_f. Sullivan (1973, equation (4.4)) formulated the identification for all irreps of SU_p and S_f.

The identification of the factors appearing in relation 5 was first given in a general form by Sullivan (1975a, equation (3.4)), although for bipartition irreps this relation underlies Kramer and Seligman's derivation of the Regge symmetries of $U_2 \supset U_1 \times U_1$ (Kramer and Seligman 1969a, equation (3.8)).

Sullivan (1975a, equation (10)) has identified the resubduction factor (his DCME) of $U_{p_1+p_2+p_3}$ with the resubduction factor (his DCME) of $S_{f_1+f_2+f_3}$. In relation 3 we have instead the appearance of the reinduction factor of $S_{f_1} \times S_{f_2} \times S_{f_3}$. The relationship between these two symmetric group transformation factors, which have the same unitary properties, involves definite phase and multiplicity choices and is not established here (see Haase and Butler 1984, § 5).

Chen (1981, equation (31)) has identified, omitting the duality factors, the coupling factors of the symmetric group with those of the unitary group (relation 4). Chen *et al* (1983) review this, but one should note that their equations (22) and (24) are inconsistent for non-simple phase irreps (see Derome (1966) or Butler (1975)),

5. Conclusions

In this paper we have shown the precise relationship between various symmetric group transformation factors and combinatorically equivalent unitary group transformation factors. We have summarised these in table 1.

Table 1. Duality relationships.

Resubduction $S_{f_1+f_2+f_3}$	$\simeq$ recoupling $(6j)$ U_p
Recoupling $(6j)$ S_f	$\simeq$ resubduction $U_{p_1p_2p_3}$
Reinduction $S_{f_1} \times S_{f_2} \times S_{f_3}$	$\simeq$ resubduction $U_{p_1+p_2+p_3}$
Coupling $(3jm)$ $S_{f_1+f_2}$	$\simeq$ coupling $(3jm)$ $U_{p_1p_2}$
Induction $S_{f_1} \times S_{f_2}$	$\simeq$ coupling $(3jm)$ $U_{p_1+p_2}$

The presence of the three duality factors complicates the dual relationship. We do not expect to be able to choose the duality factors unity even in cases without multiplicity where the factors are of modulus one. This follows from the fact that the duality factors may be related to the permutation and conjugation matrices of the various groups. Certain phase choices within the Racah–Wigner algebra of an arbitrary compact group (Butler 1975, 1981) have been made to simplify these permutation and complex conjugation symmetries of the $3jm$ and $6j$ symbols and their calculation. In particular, since the irreps of the symmetric groups are real orthogonal, all $3jm$ and $6j$ symbols can be chosen real. For the unitary groups the unity choice of the Derome–Sharp A-matrix simplifies the complex conjugation symmetry and hence the use of $3jm$ and $6j$ symbols but forces imaginary values (Bickerstaff *et al* 1982). Furthermore, the p-valued representations of U_p are isomorphic to irreps of $U_1 \times SU_p$ and $3jm$ (respectively $6j$) symbols for U_p can be factorised into a product of U_1 and SU_p $3jm$ (respectively $6j$) symbols. This depends on a factorisable phase choice which although always possible is not always used (Baird and Biedenharn 1964. So and Strottman 1979).

The determination of the duality factors is very much related to the phase choices for U_p or SU_p and S_f. Whether or not it is possible to give a simple structure to the duality factors awaits further study (Sullivan 1983).

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